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Geometry of the SU_3 Group

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A parametrization of the SU_3 group is given which is regular in the neighborhood of the unit element. The left-invariant differential forms are explicitly calculated, and a left- and right-invariant metric tensor is exhibited in these parameters.

1. INTRODUCTION

The SU_3 group appears in many fields of physics as a symmetry. It was proposed as a symmetry for strong interactions (Gell-Mann, 1962; Ne'eman, 1961). In nuclear physics it was used extensively for the classification of nuclear states (Elliott, 1963). This group gives the canonical transformations which leave the Hamiltonian of the three-dimensional harmonic oscillator invariant (Baker, 1956; Jauch and Hill, 1940). The operations of SU_3 transform a solution of the equations of motion of the oscillator into solutions of the same energy. Since the harmonic oscillator is a widely used model in physics, its symmetry group has numerous applications. Quite generally, the group appears always in quantum mechanics when three equivalent states of a physical system have to be considered together, with their possible equivalent descriptions achieved by unitary transformations.

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2. PARAMETRIZATION OF THE MANIFOLD

While even beginners in quantum mechanics know the explicit form of the transformations of the SU_2 group, the explicit transformations for SU_3 are not so well known. The usual representation of a three-dimensional special-unitary matrix,

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \tag{1}$$

by means of nine complex (18 real) variables, u_{ik}, needs ten side conditions:

$$u_{ij} u^*_{ik} = \delta_{jk}, \qquad \det U = 1 \tag{2}$$

which are quadratic and cubic in the variables. The group manifold is constructed by the intersection of nine quadratic and one cubic hypersurface in a space of 18 dimensions. Such a representation is not usable for many purposes.

Unitary matrices in any dimension can be parametrized by the exponential map or the Cayley map. The exponential map represents a unitary matrix as

$$U = e^{iH} \tag{3}$$

where H is Hermitian. A special-unitary matrix is obtained with a vanishing trace of H (Chevalley, 1946). Since this is a linear side condition, it can easily be taken into account. For SU_3 one has

$$H = \tfrac{1}{2} \lambda_A x^A \qquad (A = 1 \cdots 8) \tag{4}$$

where the λ_A are the Gell-Mann matrices, and the x^A are a set of real manifold coordinates. In Appendix A we give the explicit form of the SU_3 transformations represented by the exponential map.

Although the general properties of a Lie group can be discussed best in the exponential map, it turns out to be too unwieldy for the calculation of the invariant forms and the metric tensor.

The Cayley map represents a unitary matrix also in terms of a Hermitian matrix, H, by

$$U = (1 + iH)(1 - iH)^{-1} \tag{5}$$

A special unitary matrix (in three dimensions[2]) is obtained by the side condition:

$$\operatorname{tr} H = \det H \tag{6}$$

This is again a nonlinear side condition. The Cayley map appears also unsuitable for our calculations.

Parametrizations of SU_2 are often given in terms of Eulerian angles. This method can be generalized to higher dimension (Weyl, 1950; Murnaghan, 1962; Hermann, 1966; Wigner, 1968; Madumezia, 1971). However, explicit parametrizations of the Euler angle type suffer often from the following diseases: the coordinates in the group manifold are singular at the origin (the unit element), and the topology of the group is misrepresented. Since we want to match the Lie algebra, i.e., the tangent space, at the unit element to Gell-Mann's coordinates, we need a parametrization which is nonsingular in the neighborhood of the unit element.

This second disease is altogether incurable, since the group manifold is topologically different from the R^8. But there are still differences in morbidity for the various parametrizations. One would like to cover as much as possible of the group manifold, and one should be able to represent a fundamental toroid which consists of a Cartan subgroup, i.e., a maximal Abelian subgroup, represented by all the simultaneously diagonal elements in SU_3. This toroid ought to be described by two longitudinal angles (i.e., modulo 2π). As we have not found in the literature a suitable parametrization, we shall give one that meets our requirements.

It is proved (Epstein, 1974) that an arbitrary U_3 matrix can be factored in the form:

$$U = D \cdot U_1 \cdot U_2 \cdot U_3 \tag{7}$$

where

$$D = \operatorname{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) \tag{8a}$$

with

$$-\pi < \delta_j \leqslant \pi \qquad (j = 1,2,3) \tag{8b}$$

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_1 & \beta_1 \\ 0 & -\beta_1^* & u_1 \end{pmatrix} \tag{8c}$$

[2]For dimension $n = 2$, the condition is $\operatorname{tr} H = 0$; for n greater than 3, the condition becomes rapidly much more complex with increasing n.

$$U_2 = \begin{pmatrix} u_2 & 0 & \beta_2 \\ 0 & 1 & 0 \\ -\beta_2^* & 0 & u_2 \end{pmatrix} \tag{8d}$$

$$U_3 = \begin{pmatrix} u_3 & \beta_3 & 0 \\ -\beta_3^* & u_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8e}$$

The matrices U_j have as submatrices a particular class of SU_2 matrices described by the conditions

$$u_j = +\left(1-|\beta_j|^2\right)^{1/2} \tag{9}$$

that is,

$$u_j \text{ real with } 0 \leqslant u_j \leqslant 1 \tag{10}$$

These coordinates are regular in the neighborhood of the origin; in fact, the origin is uniquely represented by placing the β_j and the δ_j equal to zero (and thus the u_j equal to 1). These coordinates become singular (i.e., nonunique) only where the $u_j = 0$. This turns out to be where one of the diagonal elements of the original unitary matrix vanishes.

If one now writes

$$\beta_j = x_j + iy_j \tag{11}$$

one can view the nine "coordinates" of the U_3 manifold as the three x_j, the three y_j, and the three δ_j. If one desires a special unitary matrix, the condition

$$\delta_1 + \delta_2 + \delta_3 = 0 \tag{12}$$

guarantees unit determinant, but it turns out to be too restrictive, preventing the eight independent coordinates from completely covering the manifold. The condition (12) turns out to be the correct one wherever

$$-\pi \leqslant \delta_j + \delta_k \leqslant \pi \qquad (\text{all } j, k) \tag{13}$$

If, for some j, k,

$$\pi < \delta_j + \delta_k \leqslant 2\pi$$

then require

$$\sum \delta_j = 2\pi \tag{14}$$

If, for some j, k,

$$-2\pi \leqslant \delta_j + \delta_k < -\pi$$

then require

$$\sum \delta_j = -2\pi \tag{15}$$

These coordinates are useful because, as indicated in detail below, they are proportional to the "Gell-Mann" coordinates. Gell-Mann does not write down a set of coordinates for the SU_3 manifold explicitly, but they are implicit in his work, since he writes down a set of infinitesimal generators in the neighborhood of the origin which are just the λ matrices. This choice (as opposed to some linear combination) defines, by implication, an 8-leg at the origin such that the derivatives of the unitary matrix in each of the coordinate directions, evaluated at the origin, are just the λ matrices.

If we take the coordinates defined by (8) through (15), and introduce two new diagonal coordinates, α_1 and α_2, defined by

$$\delta_1 = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}, \qquad \delta_2 = -\alpha_1 + \frac{\alpha_2}{\sqrt{3}}, \qquad \delta_3 = \frac{-2\alpha_2}{\sqrt{3}} \tag{16}$$

we obtain (all derivatives evaluated at the origin)

$$\begin{array}{llll} \dfrac{\partial U}{\partial x_1} = i\lambda_7, & \dfrac{\partial U}{\partial y_1} = i\lambda_6, & \dfrac{\partial U}{\partial x_2} = i\lambda_5, & \dfrac{\partial U}{\partial y_2} = i\lambda_4 \\ \dfrac{\partial U}{\partial x_3} = i\lambda_2, & \dfrac{\partial U}{\partial y_3} = i\lambda_1, & \dfrac{\partial U}{\partial \alpha_1} = i\lambda_3, & \dfrac{\partial U}{\partial \alpha_2} = i\lambda_8 \end{array} \tag{17}$$

Note that the condition (12) for a special-unitary matrix is automatically fulfilled by the linear combination (16).[3]

The geometry of SU_3 is best exhibited in terms of (left- and right-) invariant differential forms (Chevalley, 1946). A left-invariant differential

[3]If the condition (14) is required, one can add $2\pi/3$ to each of the equations (16). If (15) is required, add $-2\pi/3$. The derivatives are unaffected.

form, $\omega(x^a)$, is defined by

$$\omega \equiv l_a(x^b)\, dx^a = l_d(x'^c)\, dx'^d \tag{18}$$

where the x'^d are related to the x^a by a left translation of the group

$$x'^d = x'^d(y^c, x^a) \qquad \text{or} \qquad x' = yx \tag{19}$$

for a fixed group element y. The definition of the right translations is obtained by interchanging y and x in (19). There are as many linearly independent left (or right) translations (with constant coefficients) as the group has parameters. We number the left-invariant forms with the index E (which runs $1 \cdots 8$, for SU_3). Thus

$$\omega^E \equiv l^E{}_a(x^b)\, dx^a \tag{20}$$

where the $l^E{}_a$ constitute a set of eight covariant vector fields which do not change their functional dependence under left translations. We distinguish upper and lower case indices for the following reasons. The indices on the coordinate differentials are manifold-tensor indices, that is, they transform with the transformation matrices of the manifold. The lower indices on the left-invariant vector fields are also tensor (transforming) indices. The upper case indices are not of this type. They do not transform at all, under a group operation. They simply label the eight different differential forms $\omega^1 \cdots \omega^8$. For example, the quantity $l^A{}_b\, dx^b$ is a manifold scalar.

The left-invariant forms for an arbitrary Lie group were given by Cartan in the exponential map (Chevalley, 1946; Cartan, 1952), but the explicit form for SU_3 becomes forbiddingly complex. For matrix groups, the following formula gives a prescription for the left-invariant differential forms. Let U be a matrix representing a group element. Then the matrix

$$\omega = U^{-1}\, dU \tag{21}$$

contains as its matrix elements left-invariant differential forms (Flanders, 1963). The ω^E are obtained from the matrix ω by

$$\omega = i\omega^E \lambda_E \tag{22}$$

This can be easily inverted, using the properties of the λ matrices to give

$$\omega^D = (1/2i)\operatorname{tr} \lambda_D \omega \tag{23}$$

3. CALCULATION OF THE INVARIANT FORMS

The calculation of the left-invariant forms now proceeds directly from (21), where U is defined by (8). We require also the following matrices:

$$
\begin{aligned}
D^{-1} &= \begin{pmatrix} e^{-i\delta_1} & 0 & 0 \\ 0 & e^{-i\delta_2} & 0 \\ 0 & 0 & e^{-i\delta_3} \end{pmatrix} \\
U_1^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_1 & -\beta_1 \\ 0 & \beta_1^* & u_1 \end{pmatrix} \\
U_2^{-1} &= \begin{pmatrix} u_2 & 0 & -\beta_2 \\ 0 & 1 & 0 \\ \beta_2^* & 0 & u_2 \end{pmatrix} \\
U_3^{-1} &= \begin{pmatrix} u_3 & -\beta_3 & 0 \\ \beta_3^* & u_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\omega = U^{-1}dU &= (DU_1U_2U_3)^{-1}[(dD)U_1U_2U_3 + D(dU_1)U_2U_3 + DU_1(dU_2)U_3 \\
&\qquad + DU_1U_2(dU_3)] \\
&= U_3^{-1}U_2^{-1}U_1^{-1}D^{-1}(dD)U_1U_2U_3 + U_3^{-1}U_2^{-1}U_1^{-1}(dU_1)U_2U_3 \\
&\quad + U_3^{-1}U_2^{-1}(dU_2)U_3 + U_3^{-1}dU_3
\end{aligned}
\tag{25}
$$

Note that $dD = D\delta$, where the matrix $\delta = \mathrm{diag}(i\,d\delta_1, i\,d\delta_2, i\,d\delta_3)$. Therefore we have

$$
\begin{aligned}
\omega &= U_3^{-1}U_2^{-1}U_1^{-1}(\delta U_1 + dU_1)U_2U_3 + U_3^{-1}U_2^{-1}(dU_2)U_3 + U_3^{-1}dU_3 \\
&\equiv \qquad A \qquad + \qquad B \qquad + \qquad C
\end{aligned}
\tag{26}
$$

Now to find the matrix $C = U_3^{-1}\,dU_3$ is straightforward:

$$
C = \begin{pmatrix} \frac{1}{2}(\beta_3\,d\beta_3^* - \beta_3^*\,d\beta_3) & u_3\,d\beta_3 - \beta_3\,du_3 & 0 \\ -u_3\,d\beta_3^* + \beta_3^*\,du_3 & -\frac{1}{2}(\beta_3\,d\beta_3^* - \beta_3^*\,d\beta_3) & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{27}
$$

Note that C is anti-Hermitian and traceless. The matrix $\omega = U^{-1}dU$ must have this property, as is well known, if U is special unitary. If U is merely unitary, the condition on the trace of ω is removed. The matrix B is also found from the definitions:

$$
\begin{aligned}
B_{11} &= \tfrac{1}{2}u_3^2(\beta_2\, d\beta_2^* - \beta_2^*\, d\beta_2)\\
B_{12} &= \tfrac{1}{2}u_3\beta_3(\beta_2\, d\beta_2^* - \beta_2^*\, d\beta_2)\\
B_{13} &= u_3(u_2\, d\beta_2 - \beta_2\, du_2)\\
B_{21} &= \tfrac{1}{2}u_3\beta_3^*(\beta_2\, d\beta_2^* - \beta_2^*\, d\beta_2)\\
B_{22} &= \tfrac{1}{2}\beta_3\beta_3^*(\beta_2\, d\beta_2^* - \beta_2^*\, d\beta_2)\\
B_{23} &= \beta_3^*(u_2\, d\beta_2 - \beta_2\, du_2)\\
B_{31} &= u_3(\beta_2^*\, du_2 - u_2\, d\beta_2^*)\\
B_{32} &= -\beta_3(u_2\, d\beta_2^* - \beta_2^*\, du_2)\\
B_{33} &= -\tfrac{1}{2}(\beta_2\, d\beta_2^* - \beta_2^*\, d\beta_2)
\end{aligned}
\tag{28}
$$

Both this and the matrix C are seen to involve the same combinations of the coordinate differentials. It will be seen later that the matrix A and also the metric tensor of the manifold involve the same combinations. It is therefore convenient to introduce the following quantities:

$$
\begin{aligned}
2a_i &= \beta_i\, d\beta_i^* - \beta_i^*\, d\beta_i\\
b_i &= u_i\, d\beta_i - \beta_i\, du_i\\
b_i^* &= u_i\, d\beta_i^* - \beta_i^*\, du_i \qquad (i = 1, 2, 3)
\end{aligned}
\tag{29}
$$

In terms of these we can write the matrices B and C as follows:

$$
C = \begin{pmatrix} a_3 & b_3 & 0 \\ -b_3^* & -a_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{30}
$$

$$
B = \begin{pmatrix} u_3^2 a_2 & u_3\beta_3 a_2 & u_3 b_2 \\ u_3\beta_3^* a_2 & \beta_3\beta_3^* a_2 & \beta_3^* b_2 \\ -u_3 b_2^* & -\beta_3 b_2^* & -a_2 \end{pmatrix}
\tag{31}
$$

The a_i are pure imaginary and the b_i^* are the complex conjugates of the b_i. From this the antihermiticity and the vanishing of the trace of the matrices B and C are easily seen. Because of the relationship (9), there must also be a relation between the a_i, b_i, and b_i^*. This is easily seen to be given by

$$
\beta_i b_i^* - \beta_i^* b_i = 2u_i a_i
\tag{32}
$$

for each value of the index i (no summation!).

A rather lengthy and tedious calculation also yields the matrix A directly from the definitions:

$$
\begin{aligned}
A_{11} &= iu_2^2u_3^2\,d\delta_1 + (u_1\beta_3 + \beta_1^*u_3\beta_2)[\beta_3^*(iu_1\,d\delta_2 + du_1) + u_3\beta_2^*(i\beta_1\,d\delta_2 + d\beta_1)] \\
&\quad + (\beta_1\beta_3 - u_1u_3\beta_2) \\
&\quad \times [\beta_3^*(i\beta_1^*\,d\delta_3 + d\beta_1^*) - u_3\beta_2^*(iu_1\,d\delta_3 + du_1)] \\
A_{12} &= iu_2^2u_3\beta_3\,d\delta_1 - (u_1\beta_3 + \beta_1^*u_3\beta_2) \\
&\quad \times [i(u_3u_1 - \beta_1\beta_3\beta_2^*)\,d\delta_2 + u_3\,du_1 - \beta_3\beta_2^*\,d\beta_1] + (\beta_1\beta_3 - u_1u_3\beta_2) \\
&\quad \times [-i(u_3\beta_1^* + \beta_3\beta_2^*u_1)\,d\delta_3 - u_3\,d\beta_1^* - \beta_3\beta_2^*\,du_1] \\
A_{21} &= iu_2^2u_3\beta_3^*\,d\delta_1 + (u_1u_3 - \beta_1^*\beta_2\beta_3^*) \\
&\quad \times [-i(\beta_3^*u_1 + \beta_1u_3\beta_2^*)\,d\delta_2 - \beta_3^*\,du_1 - u_3\beta_2^*\,d\beta_1] - (\beta_1u_3 + u_1\beta_2\beta_3^*) \\
&\quad \times [i(\beta_1^*\beta_3^* - u_1u_3\beta_2^*)\,d\delta_3 + \beta_3^*\,d\beta_1^* - u_3\beta_2^*\,du_1] \\
A_{22} &= iu_2^2\beta_3\beta_3^*\,d\delta_1 + (u_1u_3 - \beta_1^*\beta_2\beta_3^*) \\
&\quad \times [u_3(iu_1\,d\delta_2 + du_1) - \beta_3\beta_2^*(i\beta_1\,d\delta_2 + d\beta_1)] \qquad (33) \\
&\quad + (\beta_1u_3 + u_1\beta_2\beta_3^*)[u_3(i\beta_1^*\,d\delta_3 + d\beta_1^*) + \beta_3\beta_2^*(iu_1\,d\delta_3 + du_1)] \\
A_{13} &= iu_2u_3\beta_2\,d\delta_1 - (u_1\beta_3 + \beta_1^*u_3\beta_2)[u_2(i\beta_1\,d\delta_2 + d\beta_1)] \\
&\quad + (\beta_1\beta_3 - u_1u_3\beta_2)[u_2(iu_1\,d\delta_3 + du_1)] \\
A_{31} &= iu_2u_3\beta_2^*\,d\delta_1 + \beta_1^*u_2[-\beta_3^*(iu_1\,d\delta_2 + du_1) - u_3\beta_2^*(i\beta_1\,d\delta_2 + d\beta_1)] \\
&\quad + u_1u_2[\beta_3^*(i\beta_1^*\,d\delta_3 + d\beta_1^*) - u_3\beta_2^*(iu_1\,d\delta_3 + du_1)] \\
A_{23} &= iu_2\beta_2\beta_3^*\,d\delta_1 + (u_1u_3 - \beta_1^*\beta_2\beta_3^*)(iu_2\beta_1\,d\delta_2 + u_2\,d\beta_1) \\
&\quad - (\beta_1u_3 + u_1\beta_2\beta_3^*)[u_2(iu_1\,d\delta_3 + du_1)] \\
A_{32} &= iu_2\beta_2^*\beta_3\,d\delta_1 + \beta_1^*u_2[i(u_3u_1 - \beta_1\beta_2^*\beta_3)\,d\delta_2 + u_3\,du_1 - \beta_3\beta_2^*\,d\beta_1] \\
&\quad + u_1u_2[i(-u_3\beta_1^* - u_1\beta_2^*\beta_3)\,d\delta_3 - u_3\,d\beta_1^* - \beta_3\beta_2^*\,du_1] \\
A_{33} &= i\beta_2\beta_2^*\,d\delta_1 + \beta_1^*u_2^2(i\beta_1\,d\delta_2 + d\beta_1) + u_1u_2^2(iu_1\,d\delta_3 + du_1)
\end{aligned}
$$

That this is anti-Hermitian and traceless is not at all obvious. This matrix can be tremendously simplified, however, by clearing all the parentheses and rearranging the terms in the combinations suggested by (29). It will be found that the differentials in the u_i and β_i and the β_i^* will fall easily into this grouping with no terms left over. The terms in the differentials of the δ_i also fall into a regular pattern, and it is found convenient to introduce also the following abbreviations:

$$
\begin{aligned}
f_1 &= u_1 u_3 - \beta_1 \beta_2^* \beta_3 \\
f_2 &= u_1 \beta_3 + \beta_1^* \beta_2 u_3 \\
f_3 &= u_3 \beta_1 + u_1 \beta_2 \beta_3^* \\
f_4 &= \beta_1 \beta_3 - u_1 u_3 \beta_2
\end{aligned}
\tag{34}
$$

Utilizing (29) and (34) the matrix A is much more conveniently represented.

In these abbreviations the matrix A is

$$
\begin{aligned}
A_{11} &= i u_2^2 u_3^2 \, d\delta_1 + i f_2 f_2^* \, d\delta_2 + i f_4 f_4^* \, d\delta_3 + \beta_3 \beta_3^* a_1 - u_3 \beta_2 \beta_3^* b_1^* \\
&\quad - u_3^2 \beta_2 \beta_2^* a_1 + u_3 \beta_3 \beta_2^* b_1 \\
A_{12} &= i u_2^2 u_3 \beta_3 \, d\delta_1 - i \, d\delta_2 \, f_2 f_1 - i \, d\delta_3 f_4 f_3^* + u_3^2 \beta_2 b_1^* + \beta_2^* \beta_3^2 b_1 \\
&\quad - u_3 \beta_3 a_1 - \beta_2 \beta_2^* \beta_3 u_3 a_1 \\
-A_{12}^* = A_{21} &= i u_2^2 u_3 \beta_3^* \, d\delta_1 - i f_1^* f_2^* \, d\delta_2 - i f_3 f_4^* \, d\delta_3 - \beta_2 \beta_3^{*2} b_1^* - \beta_2 \beta_2^* u_3 \beta_3^* a_1 \\
&\quad - u_3 \beta_3^* a_1 - u_3^2 \beta_2^* b_1 \qquad\qquad (35) \\
A_{13} &= i u_2 u_3 \beta_2 \, d\delta_1 - i f_2 u_2 \beta_1 \, d\delta_2 + i f_4 u_1 u_2 \, d\delta_3 - \beta_3 u_2 b_1 + u_2 u_3 \beta_2 a_1 \\
-A_{13}^* = A_{31} &= i u_2 u_3 \beta_2^* \, d\delta_1 - i f_2^* \beta_1^* u_2 \, d\delta_2 + i f_4^* u_1 u_2 \, d\delta_3 + u_2 \beta_3^* b_1^* + u_2 u_3 \beta_2^* a_1 \\
A_{22} &= i u_2^2 \beta_3 \beta_3^* \, d\delta_1 + i f_1 f_1^* \, d\delta_2 + i f_3 f_3^* \, d\delta_3 - \beta_2 \beta_2^* \beta_3 \beta_3^* a_1 - \beta_2^* \beta_3 u_3 b_1 \\
&\quad + \beta_2 \beta_3^* u_3 b_1^* + u_3^2 a_1 \\
A_{23} &= i u_2 \beta_2 \beta_3^* \, d\delta_1 + i f_1^* \beta_1 u_2 \, d\delta_2 - i f_3 u_1 u_2 \, d\delta_3 + u_2 u_3 b_1 + u_2 \beta_2 \beta_3^* a_1 \\
-A_{23}^* = A_{32} &= i u_2 \beta_2^* \beta_3 \, d\delta_1 + i f_1 \beta_1^* u_2 \, d\delta_2 - i f_3^* u_1 u_2 \, d\delta_3 - u_2 u_3 b_1^* + u_2 \beta_2^* \beta_3 a_1 \\
A_{33} &= i \beta_2 \beta_2^* \, d\delta_1 + i \beta_1 \beta_1^* u_2^2 \, d\delta_2 + i u_1^2 u_2^2 \, d\delta_3 - u_2^2 a_1
\end{aligned}
$$

Remembering that the u_i are real and the a_i are pure imaginary, the anti-Hermiticity is verified by inspection, term by term. For example, for A_{11} all the terms are pure imaginary except the fifth term and the last term, which are the negative complex conjugates of each other; thus they add to give a pure imaginary result. A_{12} is easily seen to be the negative complex conjugate of A_{21}, term by term. That the trace is $i(d\delta_1 + d\delta_2 + d\delta_3)$ can be shown if one uses (9). Thus, for the SU_3 case, the trace vanishes. Thus we can write for the invariant differential forms, using (26), (30), (31), and (35), the following. We have written only the forms occurring on and above the main diagonal. The remaining are found from the anti-Hermiticity:

$$\begin{aligned}
\omega_{11} &= iu_2^2u_3^2\,d\delta_1 + if_2f_2^*\,d\delta_2 + if_4f_4^*\,d\delta_3 + \beta_3\beta_3^*a_1 - u_3\beta_2\beta_3^*b_1^* \\
&\quad - u_3^2\beta_2\beta_2^*a_1 + u_3\beta_3\beta_2^*b_1 + u_3^2a_2 + a_3 \\
\omega_{12} &= iu_2^2u_3\beta_3\,d\delta_1 - if_1f_2\,d\delta_2 - if_4f_3^*\,d\delta_3 + u_3^2\beta_2b_1^* \\
&\quad + \beta_2^*\beta_3^2b_1 - u_3\beta_3a_1 - \beta_2\beta_2^*\beta_3u_3a_1 + u_3\beta_3a_2 + b_3 \\
\omega_{13} &= iu_2u_3\beta_2\,d\delta_1 - if_2u_2\beta_1\,d\delta_2 + if_4u_1u_2\,d\delta_3 - \beta_3u_2b_1 \\
&\quad + u_2u_3\beta_2a_1 + u_3b_2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (36) \\
\omega_{22} &= iu_2^2\beta_3\beta_3^*\,d\delta_1 + if_1f_1^*\,d\delta_2 + if_3f_3^*\,d\delta_3 - \beta_2\beta_2^*\beta_3\beta_3^*a_1 \\
&\quad - \beta_2^*\beta_3u_3b_1 + \beta_2\beta_3^*u_3b_1^* + u_3^2a_1 + \beta_3\beta_3^*a_2 - a_3 \\
\omega_{23} &= iu_2\beta_2\beta_3^*\,d\delta_1 + i\beta_1u_2f_1^*\,d\delta_2 - iu_1u_2f_3\,d\delta_3 + u_2u_3b_1 \\
&\quad + u_2\beta_2\beta_3^*a_1 + \beta_3^*b_2 \\
\omega_{33} &= i\beta_2\beta_2^*\,d\delta_1 + i\beta_1\beta_1^*u_2^2\,d\delta_2 + iu_1^2u_2^2\,d\delta_3 - u_2^2a_1 - a_2
\end{aligned}$$

4. THE METRIC TENSOR

One can introduce a left- and right-invariant metric into the group manifold as follows:

For the metric tensor first we require a symmetric quadratic form in the coordinate differentials. The usual choice is to take the left-invariant metric:

$$ds^2 = \Gamma_{AB}\omega^A\omega^B \equiv \Gamma_{AB}l^A{}_c l^B{}_d\,dx^c\,dx^d \tag{37}$$

Γ_{AB} is for the moment any nonsingular set of constants, which can be taken as symmetric without any loss of generality, since $\omega^A\omega^B$ is symmetric. The form (37) is left invariant by construction. What condition will it have to obey to be also right invariant? It is well known that the metric (37) will be also invariant to right translations, if the Γ_{AB} are just the so-called Killing form, i.e.,

$$\Gamma_{AB} = K_{AB} \equiv C^E{}_{AD} C^D{}_{BE} \tag{38}$$

where the $C^E{}_{AD}$ are just the structure constants of the group.

For the SU_3 group the structure constants are found most easily from the commutation properties of the infinitesimal generators.[4] These give

$$\begin{aligned} &C^1{}_{23} = 1 \\ &C^1{}_{47} = C^2{}_{46} = C^2{}_{57} = C^3{}_{45} = -C^1{}_{56} = -C^3{}_{67} = 1/2 \\ &C^4{}_{58} = C^6{}_{78} = \sqrt{3}/2 \end{aligned} \tag{39}$$

The skew symmetry gives the other nonvanishing constants. A straightforward calculation from (38) then yields

$$K_{AB} = -3\delta_{AB} \tag{40}$$

Thus we can take as the metric tensor

$$ds^2 = \delta_{AB}\omega^A\omega^B = \sum_A \omega^A\omega^A \tag{41}$$

Since the matrix ω_{ij} is anti-Hermitian and traceless, we can write for the eight real independent forms $\omega^1 \cdots \omega^8$

$$\begin{aligned} i\omega^1 &= \omega_{11} \\ i\omega^2 &= \omega_{33} \\ i\omega^3 &= (\omega_{12} + \omega_{21})/2 \\ \omega^4 &= (\omega_{12} - \omega_{21})/2 \end{aligned} \tag{42a}$$

[4] See, for example, P. Carruthers, *Introduction to Unitary Symmetry*, pp. 49 and 50. Wiley-Interscience, New York, 1966.

$$
\begin{aligned}
i\omega^5 &= (\omega_{13} + \omega_{31})/2 \\
\omega^6 &= (\omega_{13} - \omega_{31})/2 \\
i\omega^7 &= (\omega_{23} + \omega_{32})/2 \\
\omega^8 &= (\omega_{23} - \omega_{32})/2
\end{aligned} \tag{42b}
$$

which gives

$$ds^2 = \sum \omega^A \omega^A = -(\omega_{11}\omega_{11} + \omega_{33}\omega_{33} + \omega_{12}\omega_{21} + \omega_{13}\omega_{31} + \omega_{23}\omega_{32}) \tag{43}$$

This can be explicitly calculated from (36), but the result is quite complicated and difficult to check. We can get at the metric tensor much more easily from a different expression, which we can show to be equivalent (see Appendix B).

Take

$$ds^2 = \operatorname{tr} dU \cdot dU^\dagger \tag{44}$$

This is left invariant as follows:

$$
\begin{aligned}
&U \to BU \qquad (B \text{ is a constant unitary matrix}) \\
&dU \to B\,dU \\
&dU^\dagger \to (B\,dU)^\dagger = dU^\dagger B^\dagger = dU^\dagger B^{-1} \\
&\operatorname{tr} dU \cdot dU^\dagger \to \operatorname{tr} B\,dU \cdot dU^\dagger B^{-1} = \operatorname{tr} B^{-1} B\,dU dU^\dagger = \operatorname{tr} dU \cdot dU^\dagger
\end{aligned}
$$

where we have used the cyclic property of the trace. This form of the metric tensor is also right invariant as follows:

$$
\begin{aligned}
dU &\to (dU)B \\
dU^\dagger &\to (dU \cdot B)^\dagger = B^{-1} dU^\dagger \\
\operatorname{tr} dU \cdot dU^\dagger &\to \operatorname{tr} dU \cdot BB^{-1} dU^\dagger = \operatorname{tr} dU \cdot dU^\dagger
\end{aligned}
$$

Thus both $\operatorname{tr} dU \cdot dU^\dagger$ and $K_{AB}\omega^A\omega^B$ are left- and right-invariant quadratic forms in the coordinate differentials. It is shown in Appendix B that any two forms having these properties are proportional to each other.

Consequently we can calculate the metric tensor from (44) rather than from (43), which makes the task much easier.

$$
\begin{aligned}
ds^2 &= \operatorname{tr} dU\cdot dU^\dagger = \operatorname{tr} dU\cdot dU^{-1}\\
U &= DU_1U_2U_3\\
U^{-1} &= U_3^{-1}U_2^{-1}U_1^{-1}D^{-1}\\
dU &= (dD)U_1U_2U_3 + D(dU_1)U_2U_3 + DU_1(dU_2)U_3 + DU_1U_2(dU_3)\\
dU^\dagger &= U_3^{-1}U_2^{-1}U_1^{-1}dD^{-1} + U_3^{-1}U_2^{-1}dU_1^{-1}D^{-1}\\
&\quad + U_3^{-1}dU_2^{-1}U_1^{-1}D^{-1} + dU_3^{-1}U_2^{-1}U_1^{-1}D^{-1}
\end{aligned}
$$

$$
\begin{aligned}
ds^2 = \operatorname{tr}\big[& dD\,dD^{-1} + dU_1\,dU_1^{-1} + dU_2\,dU_2^{-1} + dU_3\,dU_3^{-1}\\
&+ (dD)U_1(dU_1^{-1})D^{-1} + (dD)U_1U_2(dU_2^{-1})U_1^{-1}D^{-1}\\
&+ (dD)U_1U_2U_3(dU_3^{-1})U_2^{-1}U_1^{-1}D^{-1} + D(dU_1)U_1^{-1}(dD^{-1})\\
&+ (dU_1)U_2(dU_2^{-1})U_1^{-1} + (dU_1)U_2U_3(dU_3^{-1})U_2^{-1}U_1^{-1}\\
&+ DU_1(dU_2)U_2^{-1}U_1^{-1}(dD^{-1}) + U_1(dU_2)U_2^{-1}(dU_1^{-1})\\
&+ (dU_2)U_3(dU_3^{-1})U_2^{-1} + DU_1U_2(dU_3)U_3^{-1}U_2^{-1}U_1^{-1}(dD^{-1})\\
&+ U_1U_2(dU_3)U_3^{-1}U_2^{-1}(dU_1^{-1}) + U_2(dU_3)U_3^{-1}(dU_2^{-1})\big]
\end{aligned}
\tag{45}
$$

Surprisingly, this expression is quite straightforward to calculate. We make the following abbreviations for left- and right-invariant forms in each of the component matrices:

$$
\begin{aligned}
&D = U_0\\
&L_i = U_i^{-1}\,dU_i \qquad (i = 0\cdots 3), \qquad L_i^\dagger = (dU_i^{-1})U_i\\
&R_i = (dU_i)U_i^{-1}, \qquad R_i^\dagger = U_i\,dU_i^{-1}
\end{aligned}
\tag{46}
$$

and define $ds_i^2 = \operatorname{tr}(dU_i)(dU_i^\dagger)$.

Now if we use the cyclic property of the trace we get for (45)

$$
\begin{aligned}
ds^2 = \sum_i ds_i^2 + \operatorname{tr}\big[& L_0 R_1^\dagger + L_0^\dagger R_1 + L_1 R_2^\dagger + L_1^\dagger R_2 + L_2 R_3^\dagger + L_2^\dagger R_3 \\
& + L_0 U_1 R_2^\dagger U_1^{-1} + L_0 U_1 U_2 R_3^\dagger U_2 U_1^{-1} + L_1 U_2 R_3^\dagger U_2^{-1} \\
& + L_0^\dagger U_1 R_2 U_1^{-1} + L_0^\dagger U_1 U_2 R_3 U_2^{-1} U_1^{-1} + L_1^\dagger U_2 R_3 U_2^{-1}\big]
\end{aligned} \tag{47}
$$

Now because $U^\dagger = U^{-1}$, and because $L_i^\dagger = -L_i$ (anti-Hermiticity), and $R_i = -R_i^\dagger$, we get

$$
\begin{aligned}
ds^2 = \sum_i ds_i^2 + 2\operatorname{tr}\big[& L_0 R_1^\dagger + L_1 R_2^\dagger + L_2 R_3^\dagger + L_0 U_1 R_2^\dagger U_1^{-1} \\
& + L_1 U_2 R_3^\dagger U_2^{-1} + L_0 U_1 U_2 R_3^\dagger U_2^{-1} U_1^{-1}\big]
\end{aligned} \tag{48}
$$

Each of the submatrices (two by two) L_i is of the form (30). (30) is in fact precisely L_3. A quick calculation from the definition in (46) will yield for $R_i^\dagger$ a matrix of the same form but with opposite signs off the diagonal. For example, we have

$$
R_3^\dagger = \begin{pmatrix} a_3 & -b_3 & 0 \\ b_3^* & -a_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{49}
$$

and similarly for the indices 2 and 1. The matrix L_0 turns out to be $\operatorname{diag}(i\,d\delta_1, i\,d\delta_2, i\,d\delta_3)$. If these are substituted into (48) and we specifically restrict ourselves to SU_3 (rather than U_3) by setting $\delta_3 = -\delta_1 - \delta_2$, the metric tensor of the manifold results:

$$
\begin{aligned}
\tfrac{1}{2}\, ds^2 = {} & d\delta_1^2 + d\delta_2^2 + d\delta_1\, d\delta_2 + \sum_{i=1}^{3} \left(du_i^2 + d\beta_i\, d\beta_i^*\right) + a_1 a_2 - a_1 a_3 \\
& + a_2 a_3 - \beta_2 \beta_2^* a_1 a_3 + \beta_2^* b_3 b_1 + \beta_2 b_3^* b_1^* + i a_1 (d\delta_1 + 2\, d\delta_2) \\
& + i a_2 u_1^2 (2\, d\delta_1 + d\delta_2) + i u_1 \beta_1 \beta_2^* b_3 (d\delta_1 + 2\, d\delta_2) \\
& - i u_1 \beta_1^* \beta_2 b_3^* (d\delta_1 + 2\, d\delta_2) \\
& + i a_3 \big[(d\delta_1 - d\delta_2) + \beta_1 \beta_1^* (1 + \beta_2 \beta_2^*)(d\delta_1 + 2\, d\delta_2) \\
& \qquad - \beta_2 \beta_2^* (2\, d\delta_1 + d\delta_2)\big]
\end{aligned} \tag{50}
$$

APPENDIX A: THE EXPONENTIAL MAP FOR THE SU_3 GROUP

Let U be an element of SU_3, then $U = e^{iH}$, where $H = H^\dagger$, and $\operatorname{tr} H = 0$. We can write $H = x_j \lambda_j$, where the λ_j are the Gell-Mann matrices. The following definitions are made:

$$r^2 = x_j x_j = \tfrac{1}{2} \operatorname{tr} H^2, \qquad K = \frac{1}{r} H, \qquad \alpha = \frac{1}{3r^3} \operatorname{tr} H^3 = \tfrac{1}{3} \operatorname{tr} K^3$$

Then $U = e^{irK}$, and it can be shown that $K^n = a_n I + b_n K + c_n K^2$. Here I is the three-dimensional unit matrix. The constants a_n, b_n, c_n satisfy a recursion relation that can be written in matrix form: $\mathbf{e}_{n+1} = A\mathbf{e}_n$, or, in full,

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$$

Let g_1, g_2, g_3 stand for the eigenvalues of the matrix A, which will in general depend on α.

We have that $|\alpha| \leqslant 2/3\sqrt{3}$, and in terms of the above-defined parameters the general solution for the matrix K can be written. One must distinguish four cases:

Case 1: $\alpha = 0$: $(g_1 = 0, g_2 = 1, g_3 = -1)$. One finds

$$U = I + i \sin r(K) - (1 - \cos r) K^2$$

Case 2: $\alpha = 2/3\sqrt{3}$: $(g_1 = g_2 = -1/\sqrt{3}, g_3 = 2/\sqrt{3})$. One finds

$$\begin{aligned} U = I &\left[\frac{8}{9} e^{-ir/\sqrt{3}} + \frac{1}{9} e^{2ir/\sqrt{3}} + \frac{2}{3} \frac{ir}{\sqrt{3}} e^{-ir/\sqrt{3}} \right] \\ &+ K \left[\frac{2}{3\sqrt{3}} e^{2ir/\sqrt{3}} - \left(\frac{2}{3\sqrt{3}} - \frac{ir}{3} \right) e^{-ir/\sqrt{3}} \right] \\ &+ K^2 \left[\frac{1}{3} e^{2ir/\sqrt{3}} - \left(\frac{1}{3} + \frac{ir}{\sqrt{3}} \right) e^{-ir/\sqrt{3}} \right] \end{aligned}$$

Case 3: $\alpha = -2/3\sqrt{3}$: $(g_1 = -2/\sqrt{3}, g_2 = g_3 = 1/\sqrt{3})$. Results for this case are obtained from those for case 2 by replacing $i \to -i$, and $K \to -K$.

Case 4: $0<|\alpha|<2/3\sqrt{3}$: $(g_1 \neq g_2 \neq g_3 \neq g_1)$. In this general case the solution is expressed in terms of the eigenvalues of the matrix A:

$$U = I\left\{\frac{g_2 g_3}{h_1} e^{ig_1 r} + \frac{g_1 g_3}{h_2} e^{ig_2 r} + \frac{g_1 g_2}{h_3} e^{ig_3 r}\right\}$$

$$+ K\left\{\frac{g_1}{h_1} e^{ig_1 r} + \frac{g_2}{h_2} e^{ig_2 r} + \frac{g_3}{h_3} e^{ig_3 r}\right\}$$

$$+ K^2\left\{\frac{1}{h_1} e^{ig_1 r} + \frac{1}{h_2} e^{ig_2 r} + \frac{1}{h_3} e^{ig_3 r}\right\}$$

where we have used the abbreviations

$$h_1 = (g_1 - g_2)(g_1 - g_3)$$

$$h_2 = (g_2 - g_1)(g_2 - g_3)$$

$$h_3 = (g_3 - g_1)(g_3 - g_2)$$

Cases 1, 2, 3 can be obtained from case 4 by taking appropriate limits.

APPENDIX B

We prove here the theorem, referred to in the text, that any two metrics (symmetric quadratic forms in the coordinate differentials) which are simultaneously invariant to left and right translations are equivalent, up to a constant of proportionality, if the group is simple (the theorem is not true if the group is semisimple, a counterexample can be found).

We first prove the following:

Theorem 1. Let G be a semisimple, n-dimensional Lie group. Let

$$\omega^A = l^A{}_b(x^c)\, dx^b \tag{B1}$$

be a set of n independent left-invariant differential forms, with $A, B, a, b, \ldots = 1 \cdots n$. Define $L^a{}_B(x)$ by

$$l^A{}_b(x) L^b{}_C(x) = \delta^A{}_C \tag{B2}$$

Let

$$ds^2 = g_{ab}(x)\, dx^a\, dx^b \tag{B3}$$

be a pseudo-Riemannian metric on G, invariant under left translations. Then one can find constants $\Gamma_{AB} = \Gamma_{BA}$ such that

$$ds^2 = \Gamma_{AB}\omega^A\omega^B \tag{B4}$$

and

$$|\Gamma_{AB}| \neq 0 \tag{B5}$$

Proof. One must have

$$\Gamma_{AB}\omega^A\omega^B \equiv \Gamma_{AB}l^A{}_a l^B{}_b\, dx^a\, dx^b = g_{ab}\, dx^a\, dx^b \tag{B6}$$

One then sees immediately that this is satisfied (from B2), if

$$\Gamma_{AB} = L^a{}_A L^b{}_B g_{ab} = \Gamma_{BA} \tag{B7}$$

The determinant condition is satisfied since all the factors on the right side of (B7) have nonvanishing determinant. The constancy is inevitable since each of the Γ_{AB} is a left-invariant manifold scalar. This proves Theorem 1; we now proceed to the main theorem. ■

Theorem 2. Let G be simple, K_{AB} its Killing form, and let ds^2 be also right invariant. Then Γ_{AB} is a multiple of K_{AB}.

Proof. Since Γ_{AB} and K_{AB} are invariant under right translation and K_{AB} is nondegenerate (because G is semisimple), one can define K^{BC} by

$$K_{AB}K^{BC} = \delta_A{}^C \tag{B8}$$

and $\Gamma^C{}_A$ by

$$\Gamma^C{}_A = K^{CB}\Gamma_{BA} \tag{B9}$$

$\Gamma^C{}_A$ is also invariant under right translations, by assumption. ■

We need also the following:

Lemma. Let the matrix H_D be a generator of a right translation, then:

$$[\Gamma, H_D] = 0 \qquad (\text{all } D) \tag{B10}$$

Proof. The condition for right invariance of the metric is shown to be (Epstein, 1974, p. 46)

$$\Gamma_{AF}C^F{}_{BD} + \Gamma_{FB}C^F{}_{AD} = 0 \tag{B11}$$

where the $C^A{}_{BD}$ are the structure constants of the group. Multiply by K^{RA} to get

$$\Gamma^R{}_F C^F{}_{BD} + \Gamma^F{}_B C_F{}^R{}_D = 0 \tag{B12}$$

Using the skew symmetry of the structure constants,

$$\Gamma^R{}_F C^F{}_{BD} - C^R{}_{FD}\Gamma^F{}_B = 0 \tag{B13}$$

Since the components of the matrix Γ are $\Gamma^R{}_F$, it is obvious that we have shown that $[\Gamma, H_D] = 0$ if we take as the matrix H_D the components $(H_D)^F{}_B = C^F{}_{BD}$. Thus the Lemma is proved.

Now since the matrices H_D form an irreducible representation of the Lie algebra, and because G is simple, we must have that Γ is a multiple of the unit matrix by Schur's Lemma. Thus

$$\Gamma^C{}_A \equiv K^{CB}\Gamma_{BA} = \lambda\delta^C{}_A \tag{B14}$$

and thus

$$\Gamma_{AB} = \lambda K_{AB} \qquad (\lambda \neq 0) \tag{B15}$$

proving the theorem. ∎

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